PROBLEMS FOR THE 13^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \dots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}^n$	n-dimensional real space
S_n	set of n-permutations
$]a, b[$ and $[a, b]$	open and closed interval from a to b
$[\![a, b]\!]$	set of integers from a to b
$\lvert x\rvert$	absolute value of x
$\#M$ or $\lvert M\rvert$	cardinality of a set M
$M \times N$	Cartesian product of two sets or graphs M and N
$M \cup N, M \cap N, M \setminus N$	union, intersection and difference of two sets M and N

1. Power Sums of Distances

A set of n points $P = \{A_1, \dots, A_n\}$ is given in the real plane. For an integer $k > 0$ and a real number $c > 0$, we denote by $F_k(P, c)$ the locus of points X in the plane such that

$$|XA_1|^k + \dots + |XA_n|^k = c,$$

where $|XA_1|, \dots, |XA_n|$ are the distances from X to $A_1, \dots, A_n$ respectively.

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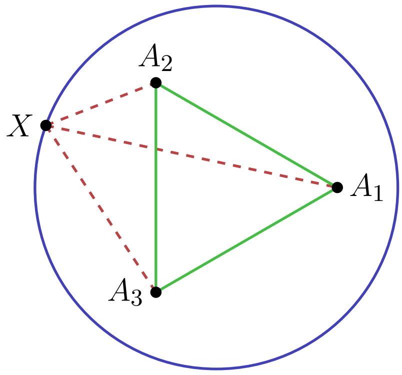

FIGURE 1. If P is the set of vertices of an equilateral triangle and c is large enough, then the loci $F_2(P, c)$ and $F_4(P, c)$ are circles.

1. Take for P the vertices of an equilateral triangle (see Figure 1).
- a) Show that $F_2(P, c)$ is either a circle or an empty set.
- b) Show that $F_4(P, c)$ is either a circle or an empty set.
- c) Describe $F_k(P, c)$ for all k.
- d) Find the smallest c for which $F_k(P, c)$ is not empty.
- e) Find all k and c such that $F_k(P, c)$ is the circumscribed circle of the triangle.

2. Consider the same questions for the vertices of an arbitrary triangle.

3. Describe $F_k(P, c)$, where P is the set of vertices of a regular n-gon.

4. In general, describe $F_k(P, c)$ for an arbitrary set P.

5. Given a circle in the plane, find the triples (n, k, c) for which there exists a set P such that $F_k(P, c)$ is this circle.

6. Generalise the problem to three dimensions.

7. Suggest and investigate additional directions of research.

2. Sequences of Coprime Integers

A finite or infinite integer sequence $a_1, a_2, \dots, a_k, \dots$ will be called *n-prime* if the numbers

$$a_1 + n, \quad a_2 + n, \quad \dots, \quad a_k + n, \dots$$

are pairwise coprime.

1. Let $a_1 < a_2$ be a sequence of two positive integers.
- a) Show that there are infinitely many $n \in \mathbb{N}$ for which the sequence is n-prime.
- b) Find necessary and/or sufficient conditions for the sequence to be n-prime for all $n \in \mathbb{N}$?

2. Let $a_1 < a_2 < a_3$ be a sequence of three positive integers.
- a) Show that there are infinitely many $n \in \mathbb{N}$ for which the sequence is n-prime.
- b) Find necessary and/or sufficient conditions for the sequence to be n-prime for all $n \in \mathbb{N}$?

3. Take $k \geq 4$. Does there exist a sequence $a_1 < a_2 < \cdots < a_k$ of k positive integers which
- a) is n-prime for all $n \in \mathbb{N}$?
- b) isn't n-prime for all $n \in \mathbb{N}$?

Start with $k = 4$.

4. In general, find or describe all finite strictly increasing sequences of positive integers that

a) are n-prime for all $n \in \mathbb{N}$.
b) are n-prime for infinitely many $n \in \mathbb{N}$.
c) aren't n-prime for all $n \in \mathbb{N}$.

5. Same questions for infinite strictly increasing sequences of positive integers.

6. A further generalisation is to consider $P(n)$-prime sequences, where $P(x) = c_d x^d + \dots + c_1 x + c_0$ is a polynomial with integer coefficients. For example, identify sequences $a_1 < a_2 < \dots < a_k$ such that $a_1 + n^{2021}, a_2 + n^{2021}, \dots, a_k + n^{2021}$ are pairwise coprime for infinitely many $n \in \mathbb{N}$.

7. Suggest and investigate other directions of research.

3. **Circular Permutations in the Plane**

Given n points in the plane $\mathbb{R}^2$, numbered from 1 to n. Assume that the points are in general position, that is, no line passes through three of them. Then there are exactly $\binom{n}{2}$ lines containing two of these points. An agent, standing anywhere in the plane outside of these lines, observes one of $(n-1)!$ circular permutations of the points (see Figure 2). A *circular n-permutation* is an arrangement of the numbers 1, …, n on a circle where only the order matters. Circular n-permutations can be presented as n-cycles $(a_1 \ \dots \ a_n)$ with $\{a_1, \dots, a_n\} = \{1, \dots, n\}$. The set of all circular n-permutations will be denoted by P_n.

We are interested in studying the following question: how many different circular permutations can the agent observe while wandering in the plane? Denote by $c(n)$ the maximum possible number of such circular n-permutations.

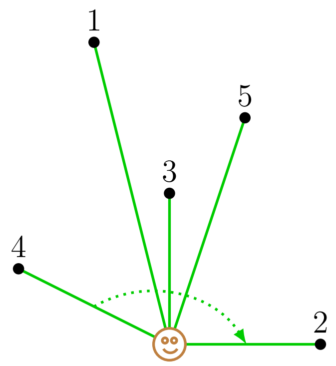

FIGURE 2. The circular permutation (4 1 3 5 2) is observed for $n = 5$.

1. The entire plane can be divided into maximal (by inclusion) connected regions such that if the agent is moving inside a region, an observed permutation doesn't change.

a) What are the boundaries of the regions?
b) How does an observed permutation change if the agent crosses a boundary?

2. Is it possible to place n points in the plane so that every circular n-permutation can be observed by the wandering agent?

3. Find lower bounds for $c(n)$ by providing explicit constructions.

4. Let $n \geq 5$ points be the vertices of a regular n-gon numbered in the clockwise direction.

a) Can the following permutation be observed:

$$(1,\ 2,\ \dots,\ m-1,\ m+1,\ \dots,\ n-1,\ n,\ m), \qquad \text{where } m = \left\lfloor \frac{n+1}{2} \right\rfloor?$$

b) Describe the permutations that can be observed.

c) Find the exact number of permutations or give upper and lower bounds.

6. Show that

$$c(n) \leq \frac{n(n-1)(n^2-n-2)}{8}.$$

7. Show that

$$c(n) \leq \frac{(n-1)(2-n)}{2} + \sum_{k=1}^{n-1}(n-k)\left(1+\frac{(k-1)(2n-k)}{2}\right).$$

8. The upper bounds in the previous two questions are equivalent to $\frac{1}{8}n^4$ as $n \to \infty$. Is it possible to improve this asymptotic?

9. Find a formula for $c(n)$.

10. A subset $S \subseteq P_n$ of circular n-permutation will be called *realisable* if there exists a configuration of n points in the plane, for which S is exactly the set of all observed permutations. Investigate realisable sets (describe properties, find possible cardinalities, etc.).

11. Suggest and study additional directions of research.

4. **Convex Functions**

Let $I \subseteq \mathbb{R}$ be a nonempty interval. A function $f : I \to \mathbb{R}$ is called *strictly convex* if, for any distinct $x, y \in I$ and any $\lambda \in]0,1[$, the following inequality is satisfied:

$$f\Big(\lambda x + (1-\lambda)y\Big) < \lambda f(x) + (1-\lambda)f(y). \tag{1}$$

Let p be a real number. A function $f : I \to]0,+\infty[$ is called *p-convex* if, for any distinct $x, y \in I$ and any $\lambda \in]0,1[$, the following inequality is satisfied:

$$f\Big(\lambda x + (1-\lambda)y\Big) < H_{\lambda,p}(f(x), f(y)),$$

where

$$H_{\lambda,p}(u,v) = \begin{cases} (\lambda u^p + (1-\lambda)v^p)^{1/p} & \text{for } p \neq 0; \\ u^\lambda v^{1-\lambda} & \text{for } p = 0. \end{cases}$$

Let $h :]0,1[\to [0,+\infty[$ be a function. A function $f : I \to \mathbb{R}$ is called *h-convex* if, for any distinct $x, y \in I$ and any $\lambda \in]0,1[$, the following inequality is satisfied:

$$f\Big(\lambda x + (1-\lambda)y\Big) < L_{\lambda,h}(f(x), f(y)),$$

where

$$L_{\lambda,h}(u,v) = h(\lambda)u + h(1-\lambda)v.$$

1. Consider the following statement (S):
"If for $a, b, c, d \in I$ one has $a + b = c + d$ and $f(a) + f(b) = f(c) + f(d)$, then $a = c$ or $a = d$."

a) Show that the statement is true for strictly convex functions.
b) Is it true for p-convex functions? Find similar true statements for p-convex functions.
c) Generalise and investigate the statement for six numbers from I instead of four.

2. Describe functions h such that the statement (S) is true for

a) all h-convex functions f.
b) some h-convex functions f, and find all such functions f.

3. Develop a method of solving equations by using convexity. For example, solve the following equations by applying the statement (S) above:

$$\sqrt[4]{x^2+x+10}+\sqrt[4]{7-x^2-x}=3,$$
$$(x^2+x+2)(x^2-3x+6)=5(2x^2-2x+3).$$

4. Take $h(\lambda)=\dfrac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}$.

a) Show that for all positive real numbers x and y and all $\lambda\in\,]0,1[$, one has the inequality
$$L_{\lambda,h}(x,y)\geqslant H_{1/2,-1}(x,y).$$

b) Find the maximum $p\in\mathbb{R}$ such that for all positive $x,y\in\mathbb{R}$ and all $\lambda\in\,]0,1[$, one has
$$L_{\lambda,h}(x,y)\geqslant H_{1/2,p}(x,y).$$

c) Find the maximum $p\in\mathbb{R}$ such that for all positive $x,y\in\mathbb{R}$ and all $\lambda\in\,]0,1[$, one has
$$L_{\lambda,h}(x,y)\geqslant H_{\lambda,p}(x,y).$$

5. Let p_0 be a real number. Suggest a way of finding a function $h:\,]0,1[\to[0,+\infty[$ such that the following inequality holds for all positive $x,y\in\mathbb{R}$, all $\lambda\in\,]0,1[$ and all $p\leqslant p_0$, but for $p>p_0$ it is false:

a) $L_{\lambda,h}(x,y)\geqslant H_{1/2,p}(x,y)$.
b) $L_{\lambda,h}(x,y)\geqslant H_{\lambda,p}(x,y)$.

6. Suggest and study additional directions of research.

5. **Odd and Even**

Let A be a set of real-valued functions defined on $\mathbb{R}$. Given a real number $r\in\mathbb{R}$, we denote by $A_O(r)$ the subset of all functions in A which are *odd at* r, that is,
$$f(r-t)+f(r+t)=0,\qquad\text{for any } t\in\mathbb{R},\tag{2}$$
and denote by $A_E(r)$ the subset of all functions in A which are *even at* r, that is,
$$f(r-t)-f(r+t)=0,\qquad\text{for any } t\in\mathbb{R}.\tag{3}$$
If B and C are two sets of real-valued functions defined on $\mathbb{R}$, then $B+C$ is the set of all functions f that can be presented as $f=g+h$ with $g\in B$ and $h\in C$.

1. Is it true that any function f can be presented as $f=g+h$, where g is an odd function at 0 and h is an even function at 0. Is such a presentation unique?

2. Check whether the relation $A=A_O(r)+A_E(r)$ is true for any $r\in\mathbb{R}$ in the following cases:

a) $A=\mathcal{F}$ is the set of all real-valued functions on $\mathbb{R}$;
b) $A=\mathcal{M}$ is the set of all bounded real-valued functions on $\mathbb{R}$;
c) $A=\mathcal{F}\setminus\mathcal{M}$ is the set of all unbounded real-valued functions on $\mathbb{R}$;
d) $A=\mathcal{D}_1$ is the set of all real-valued functions on $\mathbb{R}$ differentiable at at least one point;
e) $A=\mathcal{F}\setminus\mathcal{D}_1$ is the set of all real-valued functions on $\mathbb{R}$ which are nowhere differentiable.

3. Let $c_1,c_2>0$ be positive real numbers. Find all real $c>0$ for which there exist functions $f\in\mathcal{M}(r)$, $g\in\mathcal{M}_O(r)$ and $h\in\mathcal{M}_E(r)$ such that
$$f=g+h\quad\text{and}\quad\max_{x\in\mathbb{R}}|f(x)|=c,\quad\max_{x\in\mathbb{R}}|g(x)|=c_1,\quad\max_{x\in\mathbb{R}}|h(x)|=c_2.$$

4. Let S, S_1 and S_2 be subsets of $\mathbb{R}$. Do there exist functions $f\in\mathcal{F}(r)$, $g\in\mathcal{F}_O(r)$ and $h\in\mathcal{F}_E(r)$ such that $f=g+h$ and the sets of points where these functions are differentiable are exactly S, S_1 and S_2 respectively? Start by investigating the question for particular S, S_1 and S_2.

5. Let r_1 and r_2 be distinct real numbers. Check whether the relations $A = A_O(r_1) + A_E(r_2)$, $A = A_O(r_1) + A_O(r_2)$ and $A = A_E(r_1) + A_E(r_2)$ are true, and investigate uniqueness of a presentation $f = g + h$, in the following cases:

a) $A = \mathcal{P}_2$ is the set of polynomials of degree at most 2 with real coefficients;
b) $A = \mathcal{P}$ is the set of all polynomials with real coefficients;
c) $A = \mathcal{C}$ is the set of all continuous real-valued functions on $\mathbb{R}$;
d) $A = \mathcal{D}$ is the set of all differentiable real-valued functions on $\mathbb{R}$;
e) $A = \mathcal{M}$;
f) $A = \mathcal{F}$.

A function f is called *locally odd at* r if the equation (2) is satisfied in a neighbourhood of 0, that is, there exists $\varepsilon > 0$ such that

$$f(r-t) + f(r+t) = 0, \qquad \text{for any } t \in [-\varepsilon, \varepsilon].$$

If the equation (3) is satisfied in a neighbourhood of 0, then f is *locally even at* r. A function f is called *locally quasiodd at* r if there exists $\varepsilon > 0$ such that

$$f(r-t) + f(r+t) - 2f(r) = 0, \qquad \text{for any } t \in [-\varepsilon, \varepsilon].$$

Denote by $A_O^{loc}(r)$, $A_E^{loc}(r)$ and $A_Q^{loc}(r)$ the subsets of all functions in A which are locally odd, locally even and locally quasiodd at r respectively.

Let k be a real number. A function f is called k-*asymptotically odd at* r if

$$\lim_{t \to 0} \frac{f(r-t) + f(r+t)}{t^k} = 0,$$

and k-*asymptotically even at* r if

$$\lim_{t \to 0} \frac{f(r-t) - f(r+t)}{t^k} = 0.$$

A function f is called k-*asymptotically quasiodd at* r if

$$\lim_{t \to 0} \frac{f(r+t) + f(r-t) - 2f(r)}{t^k} = 0.$$

Denote by $A_O^k(r)$, $A_E^k(r)$ and $A_Q^k(r)$ the subsets of all functions in A which are k-asymptotically odd, k-asymptotically even and k-asymptotically quasiodd at r respectively.

6. Describe the functions $f \in A$ such that

a) $f \in A_O^{loc}(r)$ for all $r \in \mathbb{R}$;
b) $f \in A_E^{loc}(r)$ for all $r \in \mathbb{R}$;
c) $f \in A_Q^{loc}(r)$ for all $r \in \mathbb{R}$;
d) $f \in A_O^k(r)$ for all $r \in \mathbb{R}$;
e) $f \in A_E^k(r)$ for all $r \in \mathbb{R}$;
f) $f \in A_Q^k(r)$ for all $r \in \mathbb{R}$.

Do they form a linear subspace of A? If yes, what is its dimension? Start with $A = \mathcal{F}$ and $A = \mathcal{M}$.

7. Given a real k, is there a continuous function that is neither k-asymptotically odd nor k-asymptotically even at all points? Start with $k = 0$.

8. Suggest and study additional directions of research.

6. **Binomial Coefficients and Prime Numbers**

Let $S \subseteq \mathbb{N}$ be a subset of positive integers. An integer $n \geq 2$ will be called *S-compound* if, for each $1 \leq k \leq n-1$, the binomial coefficient $\binom{n}{k}$ is divisible by at least one number from S. Given a positive integer ℓ, we will say that n is *ℓ-compound* if it is S-compound for some set S consisting of exactly ℓ prime numbers.

Denote by $q(n)$ the largest prime number less than $n \geq 3$, and let $q(1) = q(2) = 0$.

1. Determine all integers $n \geq 2$ which are
  a) S-compound, where $S = \{p\}$ for a prime p;
  b) 1-compound.

2. Suppose that S consists of exactly ℓ prime numbers.
  a) Prove that there are infinitely many positive integers which are S-compound.
  b) Is it true that there are infinitely many positive integers which are not S-compound?

3. Prove that n is 2-compound in the following case:
  a) $n = p^\alpha + 1$, where p is prime and α is a non-negative integer;
  b) $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is a prime factorisation such that $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_s^{\alpha_s}$ and $n < q(n) + p_s^{\alpha_s}$;
  c) $n = p^k m$ where p is prime, m isn't divisible by p and k is large enough.

4. Let n be a positive integer having a prime factorisation $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_s^{\alpha_s}$. Prove that n is 3-compound in the following cases:
  a) n is even, it is not a power of 2, and $q(n) < n - p_s^{\alpha_s} < 2q(\frac{n}{2})$;
  b) $n < 6p_{s-1}^{\alpha_{s-1}} p_s^{\alpha_s}$;
  c) $n < 6q(\frac{n}{d})$ for a divisor $d > 1$ of n.

5. Is it true that all integers $n \geq 2$ are
  a) 2-compound?
  b) 3-compound?
  c) ℓ-compound, where $\ell > 3$ is a given integer?

6. Find or describe S-compound integers for particular finite or infinite subsets S of positive integers.

7. Generalise and investigate the problem for multinomial coefficients.

8. Suggest and study additional directions of research.

7. **Proper Numberings of Graphs**

Let $G = (V, E, \lambda)$ be a simple undirected labeled graph, where V is its vertex set, E is its edge set and $\lambda : E \to \mathbb{N}$ is a labeling of its edges with positive integers. A *proper vertex k-numbering of G* is a labeling $\nu : V \to \{1, \ldots, k\}$ of its vertices with integers from 1 to k satisfying the following condition: if any two vertices u and v are connected by an edge e, then

$$|\nu(u) - \nu(v)| \geq \lambda(e).$$

Note that it is not necessary that all the integers from 1 to k are used (see Figure 3).

For a vertex v, let $s(v)$ be the sum of the labels on all the edges with an endpoint v. Denote by $S(G)$ the maximum of such sums:

$$S(G) = \max_{v \in V} s(v).$$

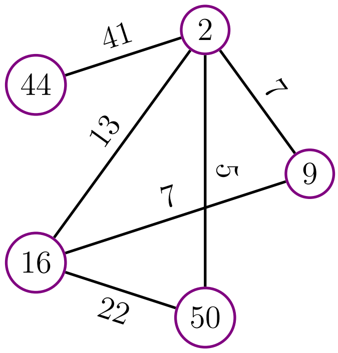

FIGURE 3. A proper vertex 50-numbering of a labeled graph with $S(G) = 66$.

1. Consider the following algorithm: list the vertices of G in an arbitrary order and, according to this order, give to each vertex the least possible number for which the required condition for a proper k-numbering will not be violated. Is it true that this algorithm successfully produces a proper vertex k-numbering with

a) $k = 2S(G) + 1$, for any G?
b) $k = 2S(G)$, for any connected G?
c) $k = 2S(G) - 1$, for any connected G with at least 2021 vertices?

2. Prove that any graph G has a proper vertex $(S(G) + 1)$-numbering.

3. Give examples of graphs G that do not have any proper vertex $S(G)$-numbering.

4. Find necessary and/or sufficient conditions for a graph G to have a proper vertex $S(G)$-numbering. First, consider the case when all edges of G are labeled with 1.

5. Find lower bounds, upper bounds or exact values (depending on G) for the minimum positive integer k such that G has a proper vertex k-numbering, if G is

a) a labeled complete graph;
b) a labeled even cycle;
c) a labeled odd cycle;
d) a labeled bipartite graph;
e) a labeled triangle-free graph;
f) a labeled planar graph.

6. Suggest and study additional directions of research.

8. Unimodality of Permutations

For a positive integer n, we denote $[\![1, n]\!] = \{1, \ldots, n\}$. Let $a_1, \ldots, a_n$ be a sequence of real numbers. It is called *strongly unimodal* if there is an index $k \in [\![1, n]\!]$ such that $a_1 < a_2 < \cdots < a_k$ and $a_k > a_{k+1} > \cdots > a_n$. It is called *unimodal* if there is a partition $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = [\![1, n]\!]$, where p and q are positive integers with $p+q = n$, such that $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$ and

$$a_{i_1} < \cdots < a_{i_p} > a_{j_1} > \cdots > a_{j_q}.$$

A sequence is called *weakly unimodal* if it is the union of a decreasing subsequence and a strongly unimodal subsequence that have the same first element. More formally, there is a partition $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = [\![2, n]\!]$, where p and q are positive integers with $p + q = n - 1$, such that $1 < i_1 < \cdots < i_p$, $1 < j_1 < \cdots < j_q$ and

$$a_1, a_{i_1}, \ldots, a_{i_p} \text{ is decreasing} \quad \text{but} \quad a_1, a_{j_1}, \ldots, a_{j_q} \text{ is strongly unimodal.}$$

Let S_n be the set of *n-permutations*, that is, the set of sequences $a_1, \ldots, a_n$ such that $\{a_1, \ldots, a_n\} = [\![1, n]\!]$. Denote by SU_n, U_n and WU_n the sets of strongly unimodal, unimodal and weakly unimodal n-permutations respectively.

A sequence is called *strongly classified* if it can be sorted in ascending order with a stack (an example is shown in Figure 4).

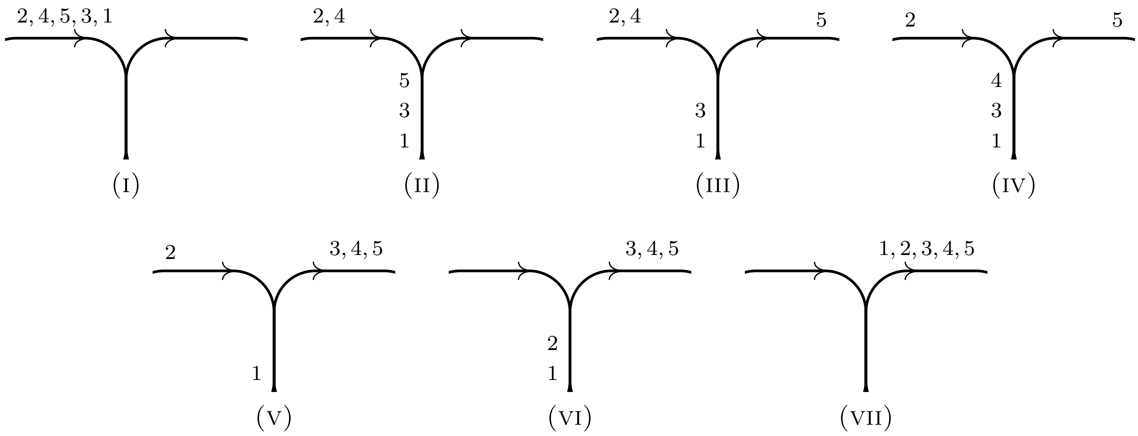

FIGURE 4. The permutation 2, 4, 5, 3, 1 is strongly classified.

A sequence is called *classified* if it can be sorted in ascending order with a limited dequeue, and *weakly classified* if it can be sorted in ascending order with a dequeue (see Figure 5).

FIGURE 5. A limited dequeue (A) and a dequeue (B).

Denote by SC_n, C_n and WC_n the sets of strongly classified, classified and weakly classified n-permutations respectively.

1. Determine for $n = 3$ and $n = 4$:
a) SU_n; b) U_n; c) WU_n; d) SC_n; e) C_n; f) WC_n.

2. Find a formula or give lower and upper bounds for the following cardinalities:
a) $|SU_n|$; b) $|U_n|$; c) $|WU_n|$; d) $|SC_n|$; e) $|C_n|$; f) $|WC_n|$.

Let $k \in \mathbb{N}$. A sequence is called *k-strongly unimodal* (resp. *k-unimodal*, *k-weakly unimodal*, *k-strongly classified*, *k-classified* or *k-weakly classified*) if it is a disjoint union of k subsequences that are strongly unimodal (resp. unimodal, weakly unimodal, strongly classified, classified or weakly classified).

For a permutation $\sigma \in S_n$, we denote by $k_{SU}(\sigma)$, $k_U(\sigma)$, $k_{WU}(\sigma)$, $k_{SC}(\sigma)$, $k_C(\sigma)$ and $k_{WC}(\sigma)$ the smallest positive integer k for which σ is k-strongly unimodal, k-unimodal, k-weakly unimodal, k-strongly classified, k-classified and k-weakly classified respectively.

For any subset $M \subseteq S_n$, define

$$k_{SU}(M) = \max_{\sigma \in M} k_{SU}(\sigma) \quad \text{and} \quad \bar{k}_{SU}(M) = \frac{1}{|M|} \sum_{\sigma \in M} k_{SU}(\sigma).$$

Similarly, we define $k_U(M)$, $k_{WU}(M)$, $k_{SC}(M)$, $k_C(M)$, $k_{WC}(M)$ and $\bar{k}_U(M)$, $\bar{k}_{WU}(M)$, $\bar{k}_{SC}(M)$, $\bar{k}_C(M)$, $\bar{k}_{WC}(M)$.

3. Find a formula and an asymptotic or give lower and upper bounds for:
a) $k_{SU}(S_n)$; b) $k_U(S_n)$; c) $k_{WU}(S_n)$; d) $k_{SC}(S_n)$; e) $k_C(S_n)$; f) $k_{WC}(S_n)$.

4. Find a formula and an asymptotic or give lower and upper bounds for:
a) $k_{SU}(SC_n)$; b) $k_{SU}(C_n)$; c) $k_{SU}(WC_n)$; d) $k_{WU}(SC_n)$; e) $k_{WU}(C_n)$; f) $k_{WU}(WC_n)$.

5. Find a formula and an asymptotic or give lower and upper bounds for:
a) $\bar{k}_{SU}(S_n)$; b) $\bar{k}_U(S_n)$; c) $\bar{k}_{WU}(S_n)$; d) $\bar{k}_{SC}(S_n)$; e) $\bar{k}_C(S_n)$; f) $\bar{k}_{WC}(S_n)$.

6. Suggest and study additional directions of research.

9. **Wobbly Tables**

We would like to place a table on the floor of a room. The floor is parametrised by a surface $z = f(x, y)$, where $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ is a continuous map. For instance, this could be

- a horizontal plane: $f_1(x, y) = 0$;
- an inclined plane: $f_2(x, y) = ax + by$ with $a, b \in \mathbb{R}$;
- a horse saddle: $f_3(x, y) = sxy$ with $s \in \mathbb{R} \setminus \{0\}$;
- a sphere: $f_4(x, y) = \sqrt{1 - \frac{x^2+y^2}{R^2}}$ with radius $R > \sqrt{2}$;
- a double periodic surface: $f_5(x, y) = \cos(\frac{2\pi x}{\omega}) \cos(\frac{2\pi y}{\omega})$ with $\omega \in]0, \infty[$.

The table has four legs of the same large length. Let $P_i = (x_i, y_i, z_i) \in [-1, 1] \times [-1, 1] \times \mathbb{R}$, where $1 \le i \le 4$, be the positions of the tips of the legs. We assume that the points P_1, P_2, P_3 and P_4 lie in the same plane and $P_1P_2P_3P_4$ is a flat convex quadrilateral $\mathcal{Q}$.

We say that the table is *grounded* if the following two conditions are met:

- the points P_1, P_2 and P_3 are situated on the floor, that is, $z_i = f(x_i, y_i)$ for $1 \le i \le 3$;
- the point P_4 is situated on the floor or above it, that is, $z_4 \ge f(x_4, y_4)$.

A grounded table is called *stabilised* if $z_4 = f(x_4, y_4)$ and *wobbly* if $z_4 > f(x_4, y_4)$.

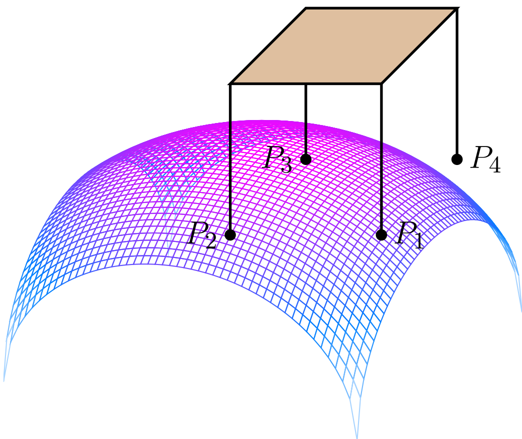

FIGURE 6. A wobbly table.

1. For the surfaces parametrised by the functions $f_1, \ldots, f_5$, can the table be grounded when

a) $\mathcal{Q}$ is a square with sides of length $\sqrt{2}$?
b) $\mathcal{Q}$ is a rectangle with sides of length r and s such that $r^2 + s^2 = 4$?
c) $\mathcal{Q}$ is a rhombus with an axis of length 2 and an axis of length ℓ, where $0 < \ell < 2$?
d) $\mathcal{Q}$ is any convex quadrilateral contained in a disc of radius 1?

2. For which of the above surfaces and quadrilaterals could the table be

a) wobbly?
b) stabilised?

3. For the above surfaces and quadrilaterals, when the table can be stabilised, what are the possible values of the angle α between the plane of the tips $(P_1P_2P_3P_4)$ and the horizontal plane $z = 0$?

4. Describe surfaces on which all tables with $\mathcal{Q}$ contained in a disc of radius 1 can be

a) grounded;
b) stabilised.

5. For a general surface $z = f(x, y)$ of the floor, give conditions on f so that the table can be stabilised for quadrilaterals of the question 1.

6. Suggest and study additional directions of research.

10. **Non-nilpotent Graphs of Groups**

Throughout this problem S_n and A_n are the symmetric and the alternating groups of degree n respectively, $D_n = \left\langle x, y \;\middle|\; x^n = y^2 = (xy)^2 = 1 \right\rangle$ is the dihedral group of order $2n$, $|G|$ is the order of a group G, $\langle x, y\rangle$ is a subgroup generated by x and y, $G_1 \simeq G_2$ means that groups G_1 and G_2 are isomorphic, $\Gamma_1 \simeq \Gamma_2$ means that graphs Γ_1 and Γ_2 are isomorphic.

Recall that a group is called *nilpotent* if any two elements of coprime order commute. Set

$$nil_G(x) = \{y \in G \mid \langle x, y\rangle \text{ is nilpotent}\} \qquad \text{and} \qquad nil(G) = \bigcap_{x \in G} nil_G(x).$$

The *non-nilpotent graph of* G, denoted by $\Gamma(G)$, is the graph whose set of vertices is $G \backslash nil(G)$ and two vertices a and b are connected by an edge if $\langle a, b\rangle$ is not nilpotent. In Figure 7, you can see the non-nilpotent graph of the symmetric group of degree 3.

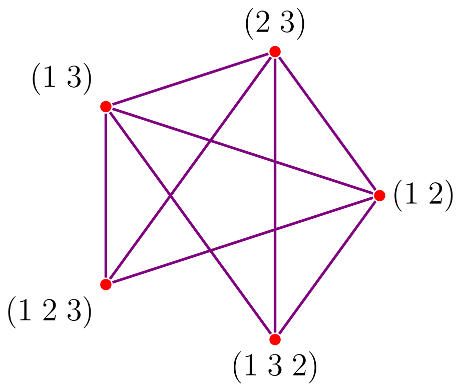

FIGURE 7. The non-nilpotent graph of S_3.

1. Consider the case that G is D_4, S_4 or A_4.

a) How many connected components does the graph $\Gamma(G)$ have?
b) If H is another finite group such that $\Gamma(H) \simeq \Gamma(G)$, is it true that $|G| = |K|$? $G \simeq K$?

2. For $G = D_n$, describe the graph $\Gamma(G)$. In particular answer the following questions.

a) How many vertices does $\Gamma(G)$ have?
b) How many edges does $\Gamma(G)$ have?
c) Is $\Gamma(G)$ connected? How many connected components does it have?

d) Does every connected component of $\Gamma(G)$ have a Hamiltonian circuit?
e) Does every connected component of $\Gamma(G)$ have an Eulerian circuit?
f) If H is another finite group such that $\Gamma(H) \simeq \Gamma(G)$, is it true that $|G| = |K|$? $G \simeq K$?

3. Same questions when G is the *generalised quaternion group* Q_{4n} of order $4n$, where $n \geq 2$:

$$Q_{4n} = \left\langle x, y \;\middle|\; x^{2n} = 1,\, x^n = y^2,\, xy = yx^{-1} \right\rangle .$$

4. Same questions when $G = S_n$.

5. Same questions when $G = A_n$.

6. Consider non-nilpotent graphs of other groups.

7. Suggest and study additional directions of research.

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